MATH 312
Lecture 5 : Agere

WHAT DOES MY MATRIX DO?
(It does whatever the Fox says it should do)

TODAY: We look at "Row Operations" from the viewpoint of Matrix Multiplication.

Three types of operations:

very common! **T1** Add "a" times the i^{th} Row to the j^{th} Row. Hopefully, $a \neq 0$.

less common, use only when pivot is zero. **T2** Interchange Row i and Row j. Hopefully, $i \neq j$

We only used this when computing inverses **T3** Scale Row i by some number b. Hopefully, "b" $\neq 0$.

Here's a matrix:

$$A = \begin{bmatrix} 3 & 1 & 7 \\ 2 & 4 & 5 \\ 1 & 0 & 3 \end{bmatrix}$$

Gaussian elimination asks, for example that we

$$\boxed{\text{ADD } -\tfrac{2}{3} R1 \text{ to } R2}$$

It turns out that there is a MATRIX (I'll call it M_1) which, when multiplied to A, executes this row operation.

In fact,

Row 2, Column 1 changed from Id.

$$M_1 = \begin{bmatrix} 1 & 0 & 0 \\ \boxed{-2/3} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

So,

$$M_1 A = \begin{bmatrix} 1 & 0 & 0 \\ -2/3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 & 1 & 7 \\ 2 & 4 & 5 \\ 1 & 0 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 1 & 7 \\ 0 & 10/3 & 1/3 \\ 1 & 0 & 3 \end{bmatrix}$$

✓ Done!

$\frac{15-14}{3}$ $\frac{1}{3}$

Now, let's say we want to do a T2 operation on the resulting matrix: exchange Row 2 and Row 3:

$$M_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

flipped row #2 and #3 of Id.

Now, $M_2 (M_1 A)$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 3 & 1 & 7 \\ 0 & 10/3 & 1/3 \\ 1 & 0 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 1 & 7 \\ 1 & 0 & 3 \\ 0 & 10/3 & 1/3 \end{bmatrix}$$

✓ Done!

okay, now let's scale that third row by 3 to make things prettier (this is a T3 operation). The matrix

$$M_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

scale the 1 in Id's third row by 3.

Clearly, $M_3 M_2 M_1 A = \begin{bmatrix} 3 & 1 & 7 \\ 1 & 0 & 3 \\ 0 & 10 & 1 \end{bmatrix}$ ✓ Done.

<u>Lesson #1</u> EVERY Row operation that comes up during Gauss or Gauss-Jordan elim. can be represented as MULTIPLICATION BY A MATRIX FROM THE LEFT.

$\begin{bmatrix} 1 & 0 & 0 \\ -2/3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix}$
M_1	M_2	M_3
$\begin{bmatrix} 1 & 0 & 0 \\ 2/3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/3 \end{bmatrix}$
(undoes the action of M_1)	(M_2 is its own inverse ∴ flip rows 2 and 3 again)	

opposite sign!

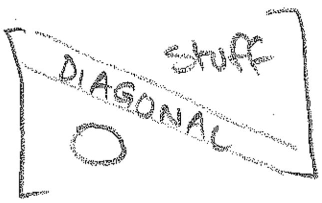
reciprocal

INVERSES

So, the point is, after <u>Gaussian</u> <u>Elimination</u>, we get something like this:

$$\underbrace{(L_k \cdots L_2 L_1)}_{\substack{\text{Some collection} \\ \text{of invertible matrices} \\ \text{coming from types} \\ \text{T1 and T2} \\ \text{(no need to scale}}} A = U$$

"Upper" triangular matrix, i.e. of the form



Since the L_i's are <u>INVERTIBLE</u>,

$$A = \underbrace{(L_1^{-1} L_2^{-1} \cdots L_k^{-1})}_{\text{order is FLIPPED!}} U$$

Call this product in parentheses "L", it is some INVERTIBLE matrix. Then,

$$A = LU$$

Lesson #2	Every matrix A is the product of an invertible matrix L and an upper triangular matrix U. And if A

requires NO ROW EXCHANGES (i.e. T2), then L can be chosen <u>LOWER</u> triangular.

In fact, we can go further! For example, if we have

> If we only do 3 row ops using pivots, $L = \begin{bmatrix} 1 & 0 & 0 \\ e_{21} & 1 & 0 \\ e_{31} & e_{32} & 1 \end{bmatrix}$

$$\bar{U} = \begin{bmatrix} 3 & 2 & 5 \\ 0 & 4 & 1 \\ 0 & 0 & 9 \end{bmatrix}, \quad \text{(upper triangular)}$$

then $\bar{U} = DU$

- D → diagonal
- U → upper triangular with 1's on diagonal!

> If $A = LU$, then
> $A\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix}$
> Becomes: $L\begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix}$, $U\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} c \\ d \end{pmatrix}$
> easy if L is lower TRIANGULAR

$$= \begin{bmatrix} 3 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 9 \end{bmatrix} \begin{bmatrix} 1 & 2/3 & 5/3 \\ 0 & 1 & 1/4 \\ 0 & 0 & 1 \end{bmatrix}.$$

So, $A = LDU$

- L → invertible (lower triangular if no T2 moves are used) ones on diagonal.
- D → diagonal
- U → upper triangular, ones on diagonal

Geometry of Matrix Multiplication!

$Id = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, Projection $= \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

Scaling: $\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}, \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$

Reflection: $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

Combine: $\begin{bmatrix} -1 & 0 \\ 0 & 3 \end{bmatrix}$

Rotation: $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ (by 90°), $\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$

Shear: $\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}.$

CHAPTER 2 OF STRANG :

- Gaussian Elimination (2x2 and 3x3)
- When do you get 0, 1 or ∞ solutions?
- Geometric Interpretation of linear systems as intersections of lines / planes
- Matrix multiplication, particularly viewing Row operations as multiplication by matrices
- Basic properties of Matrix inverses: when do they exist? How to compute them via Gauss-Jordan elimination?
- LU decomposition, using it to convert a linear system to TWO triangular systems.